The Fundamental theorem of Calculus
Lecture 5

The Short cut

Up until now we have been computing definite integrals $\int_a^b f(x)\,dx$ by solving a limit such as $\lim_{n\to\infty} R_n$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}.$$

For instance $\int_0^2 3x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} 3\left(k\frac{2}{n}\right)^2 \frac{2}{n}$

$$= \lim_{n\to\infty} 3\left(\frac{2}{n}\right)^3 \sum_{k=1}^{n} k^2 = \lim_{n\to\infty} 3\left(\frac{2}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{3\cdot 2^3\cdot 2}{6} = 2^3 = 8.$$

Early on in the development of the subject, it has been noticed however that $\int_a^b f(x)\,dx$ is intimately connected to the problem of finding anti-derivatives $\int f(x)\,dx$.
Specifically, if $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

For instance $\int 3x^2 dx = x^3 + C$ setting $F(x) = x^3$

we obtain $\int_0^2 3x^2 dx = x^3 \Big|_0^2 = 2^3 - 0^3 = 2^3 = 8.$

<u>Ex.</u> Use the integration shortcut to solve

(a) $\int_{-1}^{2} (x^3 - 2x)\, dx$

(b) $\int_1^9 \sqrt{x}\, dx$

(c) $\int_0^4 (4-t)\sqrt{t}\, dt$

(d) $\int_0^{\frac{\pi}{4}} \sec\theta \tan\theta \, d\theta$

(e) $\int_0^{\frac{\pi}{4}} \sec^2 t \, dt$

(f) $\int_1^2 \frac{v^3 + 3v^6}{v^4}\, dv$

(g) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2}\, dx$

(h) $\int_{-2}^{2} f(x)\, dx$; $f(x) =$

$$= \begin{cases} 2 & -2 \leq x \leq 0 \\ 4 - x^2 & 0 \leq x \leq 2 \end{cases}$$

Solution:

(a) $$\int_{-1}^{2} (x^3 - 2x)\,dx = \frac{1}{4}x^4 - x^2 \Big|_{-1}^{2} =$$

$\int (x^3 - 2x)\,dx$

$$= \left(\frac{1}{4}2^4 - 2^2\right) - \left(\frac{1}{4}(-1)^4 - (-1)^2\right) = -\left(\frac{1}{4} - 1\right) = \frac{3}{4}$$

(b) $$\int_{1}^{9} \sqrt{x}\,dx = \int_{1}^{9} x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} \Big|_{1}^{9} =$$

$\int x^{\frac{1}{2}}\,dx$

$$= \frac{2}{3}\,9^{\frac{3}{2}} - \frac{2}{3}\,1^{\frac{3}{2}} = 18 - \frac{2}{3}$$

(c) $$\int_{0}^{4} (4-t)\,t^{\frac{1}{2}}\,dt = \int_{0}^{4} \left(4t^{\frac{1}{2}} - t^{\frac{3}{2}}\right)dt = \frac{8}{3}t^{\frac{3}{2}} - \frac{2}{5}t^{\frac{5}{2}} \Big|_{0}^{4}$$

$\int \left(4t^{\frac{1}{2}} - t^{\frac{3}{2}}\right)dt$

$$= \frac{64}{3} - \frac{64}{5}$$

(d) $$\int_{0}^{\frac{\pi}{4}} \sec\theta \tan\theta\,d\theta = \sec\theta \Big|_{0}^{\frac{\pi}{4}} = \sqrt{2} - 1$$

$\int \sec\theta \tan\theta\,d\theta$

(e) $\int_0^{\frac{\pi}{4}} \sec^2 t\, dt = \tan t \Big|_0^{\frac{\pi}{4}} = 1$

$\int \sec^2 t\, dt$

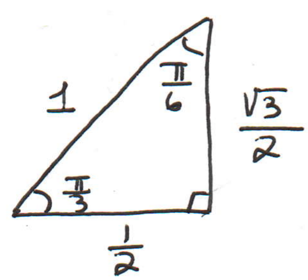
(f) $\int_1^2 \frac{v^3+3v^6}{v^4}\, dv = \int_1^2 \left(\frac{1}{v} + 3v^2\right) dv$

$\int \left(\frac{1}{v} + 3v^2\right) dv$

$= \ln|v| + v^3 \Big|_1^2 = \ln 2 + 8 - 1 = \ln 2 + 7$

(g) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2}\, dx = 8\tan^{-1} x \Big|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} = 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{4\pi}{3}$

$\int \frac{8}{1+x^2}\, dx$



(h) $\int_{-2}^{2} f(x)\, dx = \int_{-2}^{0} f(x)\, dx + \int_0^2 f(x)\, dx =$

$$= \int_{-2}^{0} 2\,dx + \int_{0}^{2} (4-x^2)\,dx = 2\cdot 2 + \left[4x - \tfrac{1}{3}x^3\right]_0^2$$

$$= 4 + 4\cdot 2 - \frac{2^3}{3} = 12 - \frac{8}{3}.$$

Ex. Calculate $\int_{-1}^{1} \frac{1}{x^2}\,dx$

Solution:

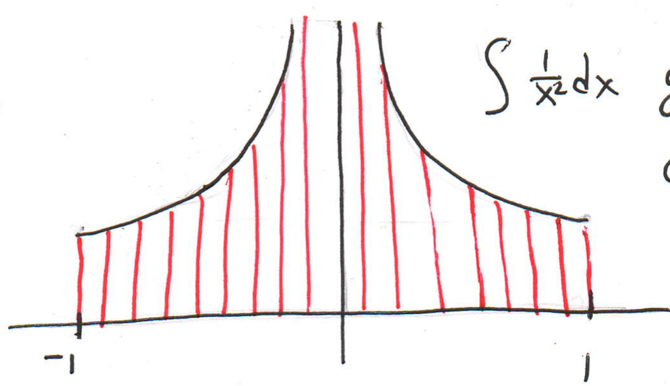
$$\int \frac{1}{x^2}\,dx = -x^{-1} + C$$

So we may be tempted to apply the shortcut:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -x^{-1}\Big|_{-1}^{1} = -\left(1^{-1} - (-1)^{-1}\right)$$

$$= -(1+1) = -2 \quad \text{Right?}$$

Wrrrong!!!



$\int \frac{1}{x^2}dx$ gives positive area!

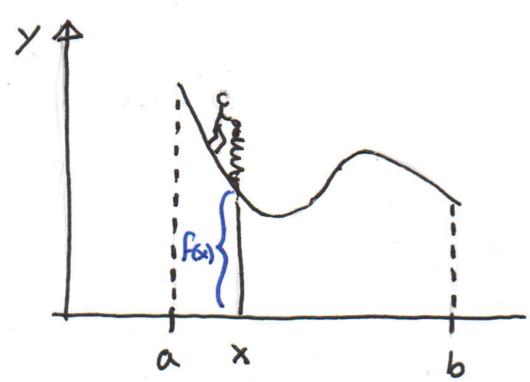
If you are using tools that you don't understand you run the risk of making major errors. There is, in fact an error in mathematical reasoning over which one person blocked me on the phone!

why did the shortcut work in some examples and didn't in another?

To see when and why the shortcut works, we need to understand the fundamental theorem of calculus,

<u>Problem:</u> Given a continuous function $y = f(x)$, how do we find the average value of this function over the interval $[a,b]$?

Imagine that you are a miner trying to estimate the average distance to bedrock



<u>Solution:</u> Take n equidistant samples from point a to point b and average them.

$x_1 = a + 1\,\frac{b-a}{n}$ - first sample point

$x_2 = a + 2\,\frac{b-a}{n}$ - second sample point

$\vdots$

$x_n = a + n\,\frac{b-a}{n}$ - n^{th} sample point.

$$\text{Ave. value estimate} = \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{1}{n}\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right).$$

Since the function f is continuous, for n large enough $f(x) \approx f(x_k)$ for all $x \in [x_{k-1}, x_k]$.

Thus it is reasonable that

$$f_{ave} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)$$

Does this limit look familiar? It should!

$$f_{ave} = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a+k\,\frac{b-a}{n}\right)\frac{1}{n}$$

$$= \lim_{n \to \infty} \frac{1}{b-a} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \frac{b-a}{n} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

<u>Ex.</u> without doing eny calculations determine

(a) $\displaystyle \lim_{h \to 0} \frac{1}{h} \int_0^h (x^2+2)\,dx$

(b) $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

$$\lim_{h \to 0} \frac{1}{h} \int_0^h f(x)\,dx$$

(c) if $f(x) = \begin{cases} \dfrac{\tan 5x}{x} & \text{if } x \neq 0 \\ 5 & \text{if } x = 0 \end{cases}$

Find $\displaystyle \lim_{h \to 0} \frac{1}{h} \int_{-h}^h f(x)\,dx$

(d) if $f(x) = \begin{cases} e^{\frac{\sin 6x}{3x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Find $\displaystyle \lim_{h \to 0} \frac{1}{h} \int_0^{3h} f(x)\,dx.$

Solution:

(a) Let $f(x) = x^2+2$. Notice that $\frac{1}{h}\int_0^h (x^2+2)\,dx$

$= \frac{1}{h}\int_0^h f(x)\,dx = f_{ave}[0,h]$. Since $f(x)$ is

continuous at 0, as h gets smaller, all the values

$f(x)$ are virtually the same as $f(0) = 0^2+2=2$.

Thus $\lim_{h\to 0} \frac{1}{h}\int_0^h f(x)dx = 2$.

(b) For $x \in [0,h]$, $\frac{\sin x}{x} \approx 1$ for h small.

Thus $\lim_{h\to 0} \frac{1}{h}\int_0^h f(x)\,dx = 1 = f(0)$.

(c) For $x\in[-h,h]$, $\frac{\tan 5x}{x} \approx 5$ for h small.

Thus $\lim_{h\to 0} \frac{1}{h}\int_{-h}^h f(x)\,dx =$

$= \lim_{h\to 0} 2\cdot\frac{1}{2h}\int_{-h}^h f(x)\,dx = 2\lim_{h\to 0} f_{ave}[-h,h]$

$= 2\cdot 5 = 10$.

(d) Notice that $\lim_{x \to 0} e^{\frac{\sin 6x}{3x}} = e^{2} \neq 0 = f(0)$.

Thus, for $x \in [0, 3h]$, $f(x) \approx e^2$ for all $x \neq 0$ when h is small.

$$\lim_{h \to 0} \frac{1}{h} \int_0^{3h} f(x)\,dx = \lim_{h \to 0} 3 \cdot \frac{1}{3h} \int_0^{3h} f(x)\,dx$$

$= 3 \lim_{h \to 0} f_{ave}[0, 3h] = 3e^2$. where $f(0)$ is an outlier (there are infinitely many values of the form $f(x) \approx e^2$ and only one $f(0) = 0$. It doesn't count!).

In the next lecture, we'll see how this helps to establish the fundamental theorem of calculus.